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Generalized Conformal Symmetries and Its Application of Hamilton Systems

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Abstract In this paper the generalized conformal symmetries and conserved quantities by Lie point transformations of Hamilton systems are studied. The necessary and sufficient conditions of conformal symmetry by the action of infinitesimal Lie point transformations which are simultaneous Lie symmetry are given. This kind type determining equations of conformal symmetry of mechanical systems are studied. The Hojman conserved quantities of the Hamilton systems under infinitesimal special transformations are obtained. The relations between conformal symmetries and the Lie symmetries are derived for Hamilton systems. Finally, as application of the conformal symmetries, an illustration example is introduced.

Keywords Conformal symmetry · Conserved quantity · Determining equation · Generalized Hamilton system

1 Introduction

The study on symmetries and conserved quantities of mechanical systems plays an important role in theory and practical value. Generally speaking, one kind of conserved quantity is presented directly or indirectly by utilizing only one symmetry in former research. Recently, the study on both symmetry and conserved quantity is flourished and therefore plentiful and substantial outcomes are achieved [1–11]. To seek conserved quantities using symmetries of dynamic systems has been a modern development direction. Its methods mainly include Noether theory [12], Lie symmetry [13] and form invariance, i.e. Mei symmetry, and so on. The corresponding conserved quantities are Noether conserved quantity [14–16], Hojman conserved quantity [17] and new type of conserved quantity [18], namely Mei conserved quantity.

Lie group theory has been used to study differential equations for a long time. It has been developed into a powerful tool to solve differential equations, to classify theory and to

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establish properties of their solution space. These aspects of Lie group theory have been described in many books and the literature [19, 20]. It is well known from Noether's theorem [21] that the constant of motion for dynamical systems can be associated with continuous transformation, which leaves the action integral invariant, of coordinates and time. Transformation that leaves the equation of motion invariant is called the Lie symmetry. Now the Lie symmetries and conserved quantities of constraint mechanical systems have been studied [22–25].

In theoretical physics, conformal symmetry is a symmetry under dilatation (scale invariance) and under the special conformal transformations. In 1997, Galiullin studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry [26]. In 2001, Robert investigated the conformal Hamiltonian systems by geometric methods, in which the author discussed the geometric structure of conformal systems with symmetry and pointed out the difference of conformal dynamics from arbitrary symmetric systems [27]. In [28–30], researchers have studied conformal invariance and conserved quantities of Lagrange systems and general holonomic systems. The author of [31] has presented the conformal invariance of non-conserved Lagrange systems with Lie point symmetry. In these years, some work has been done to discuss this problem.

In this paper, we will further study generalized conformal symmetry by Lie point transformation for Hamilton systems, and we seek out the Hojman conserved quantities of the Hamilton systems. We study the relation between the generalized conformal symmetries and the relations with Lie symmetry of Hamilton systems. Finally, as application of the conformal symmetries, we give an illustration example.

2 Diferential Equations of Generalized Hamilton Systems

The differential equation of generalized Hamilton systems with n degrees of freedom has the following forms:

$$\dot{q}_i = J_{ij}\frac{\partial H}{\partial q_j} \quad (i, j = 1, \ldots, n), \tag{1}$$

where $H = H(t, q)$ is Hamilton function, J_{ij} satisfy

$$J_{il}\frac{\partial J_{ij}}{\partial q_l} + J_{jl}\frac{\partial J_{ki}}{\partial q_l} + J_{kl}\frac{\partial J_{ij}}{\partial q_l} = 0 \tag{2}$$

and

$$J_{ij}(q) = -J_{ji}(q). \tag{3}$$

Suppose that configuration of generalized Hamilton system with n degrees of freedom are odd number, as specify a definite location motion of rigid body, Volterra equations of three kind groups and Robbins model of Lorenz equation etc.

Structure a function

$$F_i = \dot{q}_i + \Gamma_i(t, q), \tag{4}$$

where

$$\Gamma_i(t, q) = -J_{ij}\frac{\partial H}{\partial q_l}. \tag{5}$$

Research the conformal symmetry of generalized Hamilton systems, that is to seek the one-parameter group of infinitesimal transformation of (4). We take the infinitesimal transformations of time and generalized coordinate as

$$t^* = t + \Delta t, \qquad q_j^*(t^*) = q_j(t) + \Delta q_j, \tag{6}$$

their expended form

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}), \qquad q_j^*(t^*) = q_j(t) + \varepsilon\xi_j(t, \boldsymbol{q}), \tag{7}$$

where ε is the small parameter, ξ_0, ξ_j are the generators of the one-parameter infinitesimal transformation group.

Taking the infinitesimal generators as

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_j \frac{\partial}{\partial q_j}, \tag{8}$$

its first expended vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_j - \dot{q}_j \dot{\xi}_0)\frac{\partial}{\partial \dot{q}_j}. \tag{9}$$

3 Generalized Conformal Symmetry of Hamilton Systems

3.1 Definition of Conformal Symmetry and Conformal Factor

Definition In order that the Lie group of infinitesimal transformations $\xi_0(t, \boldsymbol{q})$, $\xi_i(t, q)$, if F_i satisfies the following determining equations

$$X^{(1)}(F_l) = l_l^k(F_k), \tag{10}$$

then the infinitesimal transformations is conformal symmetric. Where l_l^k is conformal factor, (10) is called determining equation.

3.2 The Necessary and Sufficient Conditions of Conformal Symmetry

To gain the general expression of conformal symmetry, we calculate the difference

$$X^{(1)}(F_i) - X^{(1)}(F_i)|_{F_i=0}. \tag{11}$$

As

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_i}{\partial q_j}\dot{q}_j \quad (i = 0, 1, \ldots, n). \tag{12}$$

Therefore

$$\begin{aligned} X^{(1)} F_i &= X^{(1)}(\dot{q}_i + \varGamma_i) \\ &= (\dot{\xi}_i - \dot{q}_i \dot{\xi}_0) + X^{(0)} \varGamma_i \\ &= \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_i}{\partial q_j}\dot{q}_j - \dot{q}_i \dot{\xi}_0 + X^{(0)} \varGamma_i \quad (i, j = 1, \ldots, n). \end{aligned} \tag{13}$$

Alike there are

$$X^{(1)}F_i|_{F_i=0} = \frac{\partial \xi_i}{\partial t} + \frac{\partial \xi_i}{\partial q_j}\alpha_j - \alpha_i \dot{\xi}_0 + X^{(0)}\Gamma_i \quad (i, j = 1, \ldots, n), \tag{14}$$

where $\alpha_j = -\Gamma_j$. From (13)–(14), we get the following equality

$$X^{(1)}(F_i) - X^{(1)}(F_i)|_{F_i=0} = \frac{\partial \xi_i}{\partial q_j}(\dot{q}_j - \alpha_j) - (\dot{q}_i - \alpha_i)\dot{\xi}_0. \tag{15}$$

On account of

$$(\dot{q}_j - \alpha_j) = \dot{q}_j + \Gamma_j = F_j. \tag{16}$$

Consequently

$$\begin{aligned} X^{(1)}(F_i) - X^{(1)}\ (F_i)|_{F_i=0} &= \frac{\partial \xi_i}{\partial q_j}F_j - F_i\dot{\xi}_0 \\ &= \left(\frac{\partial \xi_i}{\partial q_j} - \delta^i_j\dot{\xi}_0\right)F_j \quad (i, j = 1, \ldots, n). \end{aligned} \tag{17}$$

If (4) is Lie symmetry by the infinitesimal point transformation of one-parameter, thus

$$X^{(1)}(F_i)|_{F_i=0} = 0. \tag{18}$$

From the equalities of (10), (17) and (18), we get

$$(l^l_j - \beta^l_j)(F_l) = X^{(1)}(F_j)|_{F_j=0} \quad (j, l = 1, \ldots, n), \tag{19}$$

where

$$\beta^i_j = \frac{\partial \xi_i}{\partial q_j} - \delta^i_j\dot{\xi}_0 \quad (i, j = 1, \ldots, n). \tag{20}$$

So the necessary and sufficient conditions of the conformal symmetry which is simultaneous Lie symmetry by the action of the infinitesimal point transformation of one-parameter group are following

$$l^l_j = \beta^l_j. \tag{21}$$

3.3 The Criterion of Conformal Symmetry

By above analysis and discussion, we have

Proposition *For Hamilton systems* (1), *the necessary and sufficient condition of its generalized conformal symmetry which is simultaneous Lie symmetry by the action of the infinitesimal point transformation of one-parameter group are the infinitesimal generator fulfilled the following equality*

$$l^i_j = \frac{\partial \xi_i}{\partial q_j} - \delta^i_j\dot{\xi}_0 \quad (i, j = 1, \ldots, n). \tag{22}$$

4 The Conserved Quantities of Generalized Hamilton Systems

We well know that the Noether symmetry and Lie symmetry possess Noether conserved quantities. Introduce the special infinitesimal transformations of group with respect to time and coordinates in which time is not variable as

$$t^* = t, \qquad q_i^*(t^*) = q_i(t) + \varepsilon\xi_i(t, q). \tag{23}$$

If the infinitesimal generator ξ_i satisfies the following equality

$$\frac{\bar{d}}{dt}\xi_i = \frac{\partial \alpha_i}{\partial q_j}\xi_j, \tag{24}$$

where

$$\frac{\bar{d}}{dt_i} = \frac{\partial}{\partial t} + \alpha_j \frac{\partial}{\partial q_j}, \tag{25}$$

$$\alpha_j = J_{ij}\frac{\partial H}{\partial q_j}, \tag{26}$$

thus differential equation (1) is Li symmetry.

Under special infinitesimal transformation for generalized Hamilton systems (1) is conformal symmetry which is simultaneous Lie symmetry, and there exists a function $\mu = \mu(t, q)$ which satisfies following condition

$$\frac{\partial \alpha_i}{\partial q_i} + \frac{\bar{d}}{dt}\ln\mu = 0, \tag{27}$$

then Hojman conserved quantity caused directly by Lie symmetry is following form

$$I_H = \frac{1}{\mu}\frac{\partial(\mu\xi_i)}{\partial q_i} = const. \tag{28}$$

Proof in [28–30, 32].

5 Application of Conformal Symmetries

As application of the conformal symmetries of generalized Hamilton systems, an illustration example is introduced.

Let us study the conserved quantities of generalized Hamilton systems. The equation of generalized Hamilton systems can be written as the form

$$\begin{cases} \dot{q}_1 = -q_2, \\ \dot{q}_2 = q_1, \\ \dot{q}_3 = q_1 + q_2. \end{cases} \tag{29}$$

From (4) we get

$$F = \begin{pmatrix} F_1 = \dot{q}_1 + q_2 \\ F_2 = \dot{q}_2 - q_1 \\ F_3 = \dot{q}_3 - q_1 - q_2 \end{pmatrix}. \tag{30}$$

When the infinitesimal generators as

$$\begin{cases} \xi_0 = 0, \\ \xi_1 = 0, \\ \xi_2 = 0, \\ \xi_3 = \frac{1}{2}(q_1 - q_2 + q_3)^2, \end{cases} \tag{31}$$

thus

$$\begin{aligned} X^{(1)} &= \xi_0 \frac{\partial}{\partial t} + \xi_j \frac{\partial}{\partial q_j} + (\dot{\xi}_j - \dot{q}_j \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_j} \\ &= \xi_j \frac{\partial}{\partial q_j} + \dot{\xi}_j \frac{\partial}{\partial \dot{q}_j}. \end{aligned} \tag{32}$$

Consequently

$$\begin{aligned} X^{(1)} F &= \left(\xi_j \frac{\partial}{\partial q_j} + \dot{\xi}_j \frac{\partial}{\partial \dot{q}_j} \right) \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \\ &= \left(\xi_3 \frac{\partial}{\partial q_3} + \dot{\xi}_3 \frac{\partial}{\partial \dot{q}_3} \right) \begin{bmatrix} \dot{q}_1 + q_2 \\ \dot{q}_2 - q_1 \\ \dot{q}_3 - q_1 - q_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (q_1 - q_2 + q_3) & -(q_1 - q_2 + q_3) & (q_1 - q_2 + q_3) \end{bmatrix} \begin{bmatrix} \dot{q}_1 + q_2 \\ \dot{q}_2 - q_1 \\ \dot{q}_3 - q_1 - q_2 \end{bmatrix}. \end{aligned} \tag{33}$$

So we get conformal factor

$$l_j^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (q_1 - q_2 + q_3) & -(q_1 - q_2 + q_3) & (q_1 - q_2 + q_3) \end{bmatrix}. \tag{34}$$

Utilize Proposition we get

$$l_j^i = \frac{\partial \xi_i}{\partial q_j} - \delta_j^i \dot{\xi}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (q_1 - q_2 + q_3) & -(q_1 - q_2 + q_3) & (q_1 - q_2 + q_3) \end{bmatrix}. \tag{35}$$

Therefore the result is according with (34), here the systems is both conformal symmetry and Lie symmetry. From (28), when $\mu = 1$, we get the Hojman conserved quantities, as following

$$\begin{aligned} I_H &= \frac{1}{\mu} \frac{\partial(\mu \xi_i)}{\partial q_i} = \frac{\partial \xi_1}{\partial q_1} + \frac{\partial \xi_2}{\partial q_2} + \frac{\partial \xi_3}{\partial q_3} \\ &= q_1 - q_2 + q_3 = const. \end{aligned} \tag{36}$$

6 Conclusion

Hamilton system exist conformal symmetries and conserved quantities under infinitesimal Lie point transformation of generalized Hamilton systems. We can find the conformal factor in the determinate equation through the Lie symmetry. The conformal factor is conformal invariability and is simultaneously the necessary and sufficient condition of Lie symmetry by the action of infinitesimal Lie point transformations which are Lie symmetry. The conformal symmetry may let to corresponding conserved quantities when controlled some conditions.

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